Example Compute

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$$

or show it does not exist.

Solution: Maybe start by trying along $y=mx$ for some fixed $m$:

$$\lim_{x\to 0} \frac{x^3+m^3x^3}{x^2+m^2x^2} = \lim_{x\to 0} x\,\frac{1+m^3}{1+m^2} = 0$$

This does not imply the limit is 0 (but it suggests that it might be the case).

$x^2+y^2$ in denom suggests polar:

$$\lim_{\substack{r\to 0^+\\ \theta \text{ any}}} \frac{(r\cos\theta)^3+(r\sin\theta)^3}{r^2}$$

$$= \lim_{\substack{r\to 0^+\\ \theta \text{ any}}} r\,\underbrace{(\cos^3\theta+\sin^3\theta)}_{\text{this is between } -2 \text{ and } 2}$$

$$-2r \le r(\cos^3\theta+\sin^3\theta) \le 2r$$

these $\to 0$

So by Squeeze thm we have

$$\lim_{\substack{r\to 0^+\\ \theta \text{ any}}} r(\cos^3\theta+\sin^3\theta) = 0.$$

$$\lim_{(x,y)\to(0,0)} \frac{x^7y^6 + x^3 + y^3 + x^2y^2 + y^4}{x^2+y^2}$$

$$= 0$$

Same strategy works. Key point:

$$\frac{r^{13}\cos^7\theta \sin^6\theta}{r^2}$$

Squeeze... $\longrightarrow 0$.